Written Homework 2

① Working from home:

$\frac{2}{3}(10) = \frac{20}{3}$ hours on research

$\frac{1}{6}(10) = \frac{10}{6}$ hours teaching

$\frac{1}{2}(10) = 5$ cups of coffee

Padelford:

$\frac{1}{3}(15) = 5$ hours on research

$\frac{1}{2}(15) = \frac{15}{2}$ hours teaching

$0(15) = 0$ cups of coffee

Allegro:

$\frac{5}{12}(2) = \frac{5}{6}$ hours research

$\frac{5}{12}(2) = \frac{5}{6}$ hours teaching

$1(2) = 2$ cups of coffee

Research (total) $= \frac{75}{6}$ hours/week

Teaching (total) $= \frac{70}{6}$ hours/week

coffee (total) $= 7$ cups/week

(a) Total stuff accomplished $= 10\begin{bmatrix} 2/3 \\ 1/6 \\ 1/2 \end{bmatrix} + 15\begin{bmatrix} 1/3 \\ 1/2 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 5/12 \\ 5/12 \\ 1 \end{bmatrix} = \begin{bmatrix} 75/6 \\ 70/6 \\ 7 \end{bmatrix}$

$$\left.\begin{aligned} \tfrac{2}{3}t_1 + \tfrac{1}{3}t_2 + \tfrac{5}{12}t_3 &= 15 \\ \tfrac{1}{6}t_1 + \tfrac{1}{2}t_2 + \tfrac{5}{12}t_3 &= 10 \\ \tfrac{1}{2}t_1 + 0t_2 + 1t_3 &= 11 \end{aligned}\right\} \left[\begin{array}{ccc|c} \frac{2}{3} & \frac{1}{3} & \frac{5}{12} & 15 \\ \frac{1}{6} & \frac{1}{2} & \frac{5}{12} & 10 \\ \frac{1}{2} & 0 & 1 & 11 \end{array}\right] \begin{aligned} R_1 &= \tfrac{2}{3}R_1 \\ R_2 &= -\tfrac{1}{6}R_1 + R_2 \\ R_3 &= -\tfrac{1}{2}R_1 + R_3 \end{aligned} \Rightarrow \sim \left[\begin{array}{ccc|c} 1 & 1/2 & 5/8 & 45/2 \\ 0 & 5/12 & 5/16 & 25/4 \\ 0 & -1/4 & 11/16 & -1/4 \end{array}\right]$$

$$\begin{aligned} R_2 &= \tfrac{5}{12}R_2 \\ R_3 &= \tfrac{1}{4}R_2 + R_3 \\ R_1 &= -\tfrac{1}{2}R_2 + R_1 \end{aligned} \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/4 & 15 \\ 0 & 1 & 3/4 & 15 \\ 0 & 0 & 7/8 & 7/2 \end{array}\right] \begin{aligned} R_3 &= \tfrac{7}{8}R_3 \\ R_2 &= -\tfrac{3}{4}R_3 + R_2 \\ R_1 &= -\tfrac{1}{4}R_3 + R_1 \end{aligned} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 14 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

So, 14 hours at home, 12 hours at Padelford, 4 hours at Allegro (b)

(c) Vector equation: $t_1\begin{bmatrix} 2/3 \\ 1/6 \\ 1/2 \end{bmatrix} + t_2\begin{bmatrix} 1/3 \\ 1/2 \\ 0 \end{bmatrix} + t_3\begin{bmatrix} 5/12 \\ 5/12 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \\ 11 \end{bmatrix}$

Matrix eqn $A\vec{t} = \vec{w}$: $\begin{bmatrix} 2/3 & 1/3 & 5/12 \\ 1/6 & 1/2 & 5/12 \\ 1/2 & 0 & 1 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \\ 11 \end{bmatrix}$

In this context, t_1 represents time worked at home, t_2 represents time worked at Padelford, and t_3 represents time worked at the coffeeshop. $\vec{w}$ represents the total output of research (hours), teaching (hours), and coffee consumed (cups) per week.

(d) Lounge: $t_4 \begin{bmatrix} 1/2 \\ 1/3 \\ 1/3 \end{bmatrix} \Rightarrow$ just by comparing the components of this vector to the other ones, we can see that, per unit time, working in the lounge gives a greater research benefit compared to Padelford or the coffeeshop, gives a greater teaching benefit compared to working from home, and gives a greater coffee consumption benefit compared to Padelford. So Vasu is partially correct, since it depends on what aspect Jake wants to maximize.

Written Homework 2 (continued)

② $x = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 6 \end{bmatrix} + s\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -1-s \\ 1+2s \\ s \\ 6 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$\left.\begin{aligned} x_1 + s &= -1 \\ x_2 - 2s &= 1 \\ x_3 - s &= 0 \\ x_4 &= 6 \end{aligned}\right\} s = x_3 \Rightarrow \begin{aligned} x_1 + x_3 &= -1 \\ x_2 - 2x_3 &= 1 \\ 0 &= 0 \\ x_4 &= 6 \end{aligned}$$

In matrix form:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix} \Rightarrow \text{So, } A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

③ Not $\mathbb{R}^3 \Rightarrow$ make $\langle -4, z_1, z_2 \rangle$ a linear combination of $(2,-1,3)$ and $(1,2,2)$

$$\begin{vmatrix} 2 & 1 & -4 \\ -1 & 2 & z_1 \\ 3 & 2 & z_2 \end{vmatrix} = 0 \quad \begin{aligned} &2(2z_2 - 2z_1) - 1(-z_2 - 3z_1) - 4(-8) = 0 \\ &5z_2 - z_1 + 32 = 0 \Rightarrow \boxed{z_1 = 5z_2 + 32} \text{ (for any value of } z_1) \end{aligned}$$

④ $$\begin{bmatrix} 1 & -3 & 1 & 4 \\ 2 & 0 & -8 & -2 \\ -6 & 6 & a & b \end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 + 6R_1 \end{matrix} \sim \begin{bmatrix} 1 & -3 & 1 & 4 \\ 0 & 6 & -10 & -10 \\ 0 & -12 & a+6 & b+24 \end{bmatrix} \begin{matrix} R_3 = R_3 + 2R_2 \\ \sim \end{matrix} \begin{bmatrix} 1 & -3 & 1 & 4 \\ 0 & 6 & -10 & -10 \\ 0 & 0 & a-14 & b+4 \end{bmatrix}$$

(a) The system has infinitely many solutions when $a = 14$ and $b = -4$

(b) The system has 1 solution when

(c) System has no solution when $a = 14$ & $b \neq -4$

⑤ $P = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : 2x_1 - x_2 + 4x_3 = 0 \right\} \Rightarrow$ equation of plane in $\mathbb{R}^3$

$$x_1 = \frac{x_2}{2} - 2x_3 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{x_2}{2} - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}}_{u_1} + x_3 \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{u_2}$$

(a) $\boxed{\text{so: } \vec{u}_1 = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} \text{ \& } \vec{u}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}$

$$\begin{bmatrix} 2 & 3 & -5 & b_1 \\ -7 & 2 & 8 & b_2 \\ -1 & 1 & -5 & b_3 \end{bmatrix} \begin{matrix} R_2 = 7R_1 - 2R_2 \\ R_3 = R_1 + 2R_3 \end{matrix} \sim \left[\begin{array}{ccc|c} 2 & 3 & -5 & b_1 \\ 0 & 17 & -51 & 7b_1 - 2b_2 \\ 0 & 5 & -15 & b_1 + 2b_3 \end{array}\right] R_3 = \frac{1}{17}R_2 - \frac{1}{5}R_3$$

$$\sim \left[\begin{array}{ccc|c} 2 & 3 & -5 & b_1 \\ 0 & 1 & -3 & \frac{7b_1 - 2b_2}{17} \\ 0 & 0 & 0 & \frac{7b_1 - 2b_2}{17} - \frac{b_1 + 2b_3}{5} \end{array}\right] \quad \text{Now, } \frac{7b_1 - 2b_2}{17} - \frac{b_1 + 2b_3}{5} = 0$$

which can be rewritten in plane form as: $\frac{18}{85}b_1 - \frac{2}{17}b_2 - \frac{2}{5}b_3 = 0$, which is the equation of a plane in $\mathbb{R}^3$. (b)

Written Homework 2

(1) When Jake works from home, he typically spends 40 minutes of each hour on research, and 10 on teaching, and drinks half a cup of coffee. (The remaining time is spent on the internet.) For each hour he works in the math department, he spends around 20 minutes on research and 30 on teaching, and doesn't drink any coffee. Lastly, if he works at a coffeeshop for an hour, he spends 25 minutes each on research and teaching, and drinks a cup of coffee.

(**Note:** be careful about units of minutes versus hours.)

(a) Last week, Jake spent 10 hours working from home, 15 hours working in his office in Padelford Hall, and 2 hours working at Cafe Allegro. Compute what was accomplished, and express the result as a vector equation.

(b) This week, Jake has 15 hours of research to work on and 10 hours of work related to teaching. He also wants 11 cups of coffee, because... of... very important reasons. How much time should he spend working from home, from his office, and from the coffeeshop?

(c) Describe the situation in part (b) as a vector equation and a matrix equation $A\vec{t} = \vec{w}$. What do the vectors $\vec{t}$ and $\vec{w}$ mean in this context? For which other vectors $\vec{w}$ does the equation $A\vec{t} = \vec{w}$ have a solution?

(d) Jake tries working in the math department lounge for an hour, and gets 30 minutes of research and 20 minutes of teaching work done, while having time to drink $\frac{1}{3}$ of a cup of coffee. Not bad. But Jake's colleague Vasu claims that there's no need to work in the lounge – the other options already give enough flexibility. Is he right? Explain mathematically.

(2) Find a 3×4 matrix A, in *reduced* echelon form, with free variable x_3, such that the general solution of the equation $A\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}$ is

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 6 \end{bmatrix} + s \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix},$$

where s is any real number.

(3) Find all values z_1 and z_2 such that $(2, -1, 3)$, $(1, 2, 2)$, and $(-4, z_1, z_2)$ do not span $\mathbb{R}^3$.

(4) Consider the following linear system with a and b unknown non-zero constants.

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & x_3 & = & 4 \\ 2x_1 & & & - & 8x_3 & = & -2 \\ -6x_1 & + & 6x_2 & + & ax_3 & = & b \end{array}$$

(a) For what values of a and b does the system have infinitely many solutions?
(b) Given an example of a and b where the system has exactly one solution.
(c) Give an example of a and b for which the system has no solutions.

(5) (a) The set

$$P = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : 2x_1 - x_2 + 4x_3 = 0 \right\}$$

is a plane in $\mathbb{R}^3$. Find two vectors $\mathbf{u}_1,\ \mathbf{u}_2 \in \mathbb{R}^3$ so that $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\} = P$. Explain your answer.

(b) Consider the three vectors $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$. Let $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^3$. Use Gaussian elimination to determine which vectors $\mathbf{b}$ are in $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.
Without further calculation, conclude that $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a plane in $\mathbb{R}^3$ and identify an equation of the plane in the form $ax_1 + bx_2 + cx_3 = 0$.